

Classifications of quasiseparable matrices in terms of recurrence relations

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Orthogonal Polynomials Related to Structured Matrices

Moment Matrices

➡ **Hankel matrices.** Defined by $\mathcal{O}(n)$ parameters $\{h_k\}$.

$$H = \left[h_{k+j} \right] = \begin{bmatrix} h_0 & \mathbf{h}_1 & h_2 & \cdots & h_{n-1} \\ \mathbf{h}_1 & h_2 & & \ddots & \vdots \\ h_2 & & \ddots & & h_{2n-3} \\ \vdots & \ddots & & h_{2n-3} & \mathbf{h}_{2n-2} \\ h_{n-1} & \cdots & h_{2n-3} & \mathbf{h}_{2n-2} & h_{2n-1} \end{bmatrix}$$

➡ **Toeplitz matrices.** Defined by $\mathcal{O}(n)$ parameters $\{t_k\}$.

$$C = \left[t_{k-j} \right] = \begin{bmatrix} t_0 & \mathbf{t}_{-1} & \cdots & \cdots & t_{-n+1} \\ t_1 & t_0 & \mathbf{t}_{-1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & t_0 & \mathbf{t}_{-1} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{bmatrix}$$

Orthogonal Polynomials Related to Structured Matrices

Moment Matrices

- ▣ Both of these classes of matrices are related to **orthogonal polynomials**.
- ▣ For a given inner product, the **moment matrix** is

$$M = [\langle x^k, x^j \rangle] = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle & \langle 1, x^2 \rangle & \dots & \langle 1, x^n \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle & \langle x, x^2 \rangle & \dots & \langle x, x^n \rangle \\ \langle x^2, 1 \rangle & \langle x^2, x \rangle & \langle x^2, x^2 \rangle & \dots & \langle x^2, x^n \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ \langle x^n, 1 \rangle & \langle x^n, x \rangle & \langle x^n, x^2 \rangle & \dots & \langle x^n, x^n \rangle \end{bmatrix}$$

- ▣ For an inner product defined by integration on the **real line**,

$$\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)w^2(x)dx, \quad \Rightarrow \quad \langle x^k, x^j \rangle = \int_a^b x^{(k+j)}w^2(x)dx,$$

and M is **Hankel**.

- ▣ **Hankel** matrices are related to **real-orthogonal polynomials**.

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- For an inner product defined by integration on the **unit circle**,

$$\langle p(x), q(x) \rangle = \int_{-\pi}^{\pi} p(e^{i\theta}) \cdot \overline{q(e^{i\theta})} w^2(\theta) d\theta \Rightarrow \langle x^k, x^j \rangle = \int_{-\pi}^{\pi} x^{(k-j)} w^2(\theta) d\theta,$$

and M is **Toeplitz**.

- Toeplitz** matrices are related to **Szegő polynomials**.

Orthogonal Polynomials Related to Structured Matrices

Recurrent Matrices

➡ **Tridiagonal matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

➡ **Unitary Hessenberg matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$U = \begin{bmatrix} -\rho_1 \rho_0^* & -\rho_2 \mu_1 \rho_0^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_1 \rho_0^* \\ \mu_1 & -\rho_2 \rho_1^* & \cdots & -\rho_n \mu_{n-1} \cdots \mu_2 \rho_1^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n \mu_{n-1} \rho_{n-2}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n \rho_{n-1}^* \end{bmatrix}$$

Orthogonal Polynomials Related to Structured Matrices

Recurrent Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- The system of polynomials defined by $\mathbf{r}_k(x) = \det(xI - T)_{(k \times k)}$ where

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

are real-orthogonal polynomials.

Orthogonal Polynomials Related to Structured Matrices

Recurrent Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- The system of polynomials defined by $\mathbf{r}_k(x) = \det(xI - U)_{(k \times k)}$ where

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are Szegő polynomials.

Generalizations of these Structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	????????????????????????????????

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Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	quasiseparable matrices

Quasiseparable Matrices

- **Definition.** A matrix C is (H, m) -**quasiseparable** if it is strongly upper Hessenberg (nonzero subdiagonals, zeros below that) and

$$\max \text{Rank} C_{12} = m$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline & * \end{array} \right]$$

- **Previous work.** Bini, Chandrasekaran, Eidelman, Fasino, Gemignani, Gohberg, Gu, Kailath, Koltracht, Mastronardi, Olshevsky, Tyrtyshnikov, Van Barel, Vandebril...
- A system of polynomials related to an (H, m) -quasiseparable matrix C as **characteristic polynomials** of principal submatrices of C , i.e.

$$r_k(x) = \det(xI - C_{k \times k})$$

will be called (H, m) -**quasiseparable polynomials**.

Important Special Cases of Quasiseparable Matrices

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C are **real orthogonal polynomials** with recurrence relations

$$r_k(x) = \frac{1}{q_k}(x - d_k)r_{k-1}(x) - \frac{g_{k-1}}{q_k}r_{k-2}(x)$$

- The matrix C is **$(H, 1)$ -quasiseparable**.

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Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

► The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C are the **Szegő polynomials** with recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ xr_{k-1}(x) \end{bmatrix}$$

► The matrix C is $(H, 1)$ -**quasiseparable**.

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The Difference Between These Motivating Examples

Unitary Hessenberg matrices.

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 \end{bmatrix}$$

Tridiagonal matrices.

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

The Difference Between These Motivating Examples

➡ **Unitary Hessenberg matrices.** strictly upper triangular part is part of a low rank matrix.

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 \\ \frac{-\rho_1^* \rho_1}{\mu_1} & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 \\ -\rho_2^* \rho_1 & \frac{-\rho_2^* \rho_2}{\mu_2} & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 \\ \frac{\mu_1 \mu_2}{-\rho_3^* \rho_1} & \frac{\mu_2}{-\rho_3^* \rho_2} & \frac{-\rho_3^* \rho_3}{\mu_3} & -\rho_3^* \rho_4 \\ \mu_1 \mu_2 \mu_3 & \mu_2 \mu_3 & \mu_3 & \mu_3 \end{bmatrix}$$

➡ **Tridiagonal matrices.**

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

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 -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 \\
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 \frac{-\rho_3^* \rho_1}{\mu_1 \mu_2 \mu_3} & \frac{-\rho_3^* \rho_2}{\mu_2 \mu_3} & \frac{-\rho_3^* \rho_3}{\mu_3} & -\rho_3^* \rho_4
 \end{bmatrix}$$

➡ **Tridiagonal matrices.** strictly upper triangular part is NOT part of a low rank matrix.

$$C = \begin{bmatrix}
 & g_1 & 0 & 0 & 0 \\
 & & g_2 & 0 & 0 \\
 & & & g_3 & 0 \\
 & & & & g_4
 \end{bmatrix}$$

Semiseparable matrices

▣▣▣ **Definition.** A matrix R is called (r_L, r_U) -semiseparable if for some r_L, r_U we have

$$R = D + \text{tril}(R_L) + \text{triu}(R_U),$$

where $\text{rank} R_L = r_L$, $\text{rank} R_U = r_U$, with some R_L, R_U .

▣▣▣ **Example.** $(1, 1)$ -semiseparable:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

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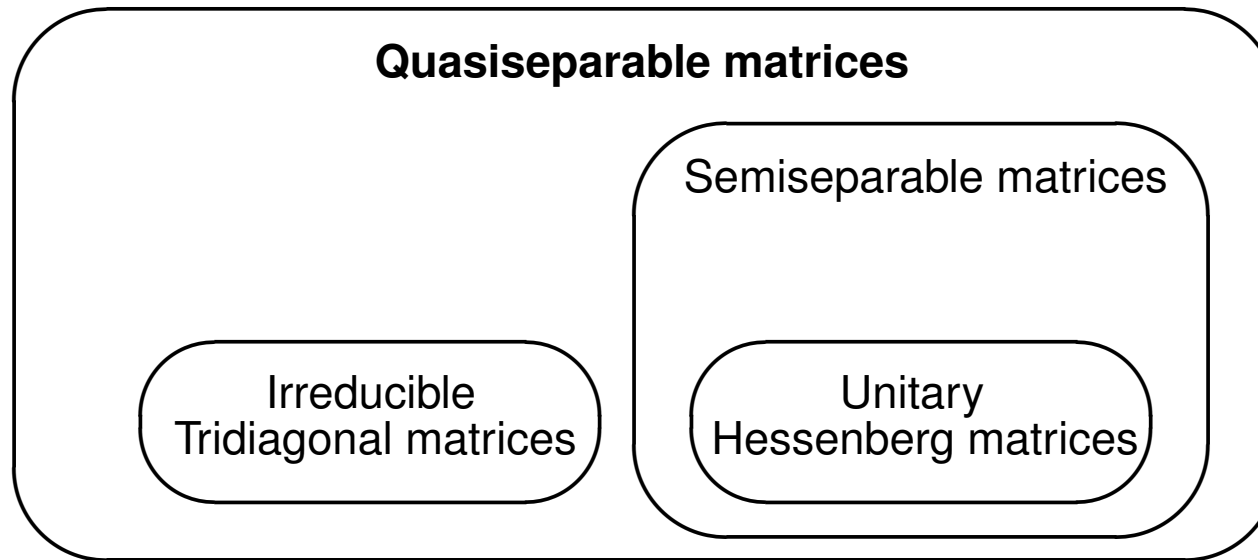
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$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Quasiseparable, Semiseparable, and Subclasses



Generator Representation of an $(H, 1)$ -Quasiseparable Matrix

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

Generator Representation of an $(H, 1)$ -Quasiseparable Matrix

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

► This **generator representation** exists for any $(H, 1)$ -**quasiseparable** matrix.

Classification of $(H, 1)$ -quasiseparable matrices in terms of recurrence relations

Joint work with Yuli Eidelman, Israel Gohberg, and Vadim Olshevsky

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

$$\mathbf{r}_k(x) = \det(xI - A)_{(\mathbf{k} \times \mathbf{k})}$$

Efficient Recurrence Relations for Quasiseparable Polynomials

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Recurrence relations

$$\mathbf{r}_k(x) = \mathbf{x} \cdot \mathbf{r}_{k-1}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

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Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

$$\mathbf{r}_k(x) = 2\mathbf{x} \cdot \mathbf{r}_{k-1}(x) - \mathbf{r}_{k-2}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
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Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

$$\mathbf{r}_k(x) = (\alpha_k \mathbf{x} - \delta_k) \mathbf{r}_{k-1}(x) - \gamma_k \mathbf{r}_{k-2}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
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Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations (2-term)

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{r}_k(x) \end{bmatrix}$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations (3-term)

$$\mathbf{r}_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) \mathbf{r}_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) \mathbf{r}_{k-2}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real-orthogonal polynomials
Unitary Hessenberg matrix	Szegő polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

????????????????????

Three-term Recurrence Relations.

Consider the class of polynomials satisfying more general three-term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - (\beta_k x + \gamma_k)r_{k-2}(x)$$

▮▮▮ Real-orthogonal polynomials: $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

▮▮▮ Szegő polynomials (orthogonal on the unit circle): $\gamma_k = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) r_{k-2}(x)$$

Three-term Recurrence Relations.

Consider the class of polynomials satisfying more general three-term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - (\beta_k x + \gamma_k)r_{k-2}(x)$$

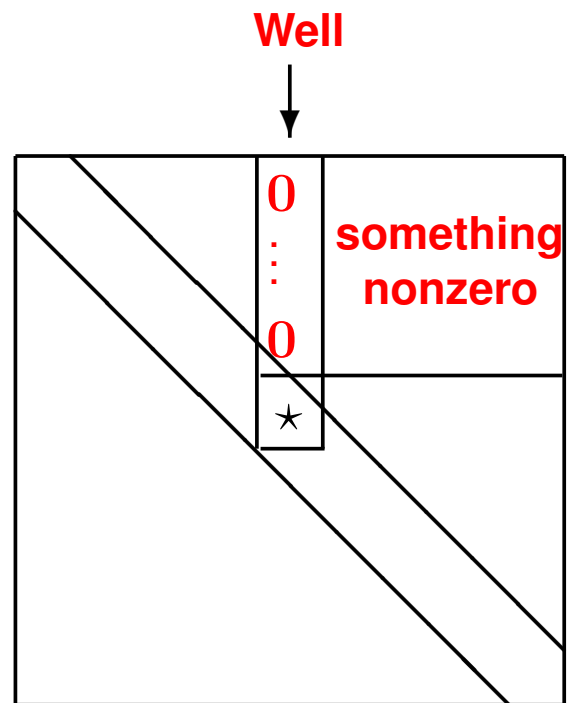
▮▮▮ Real-orthogonal polynomials: $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k)r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

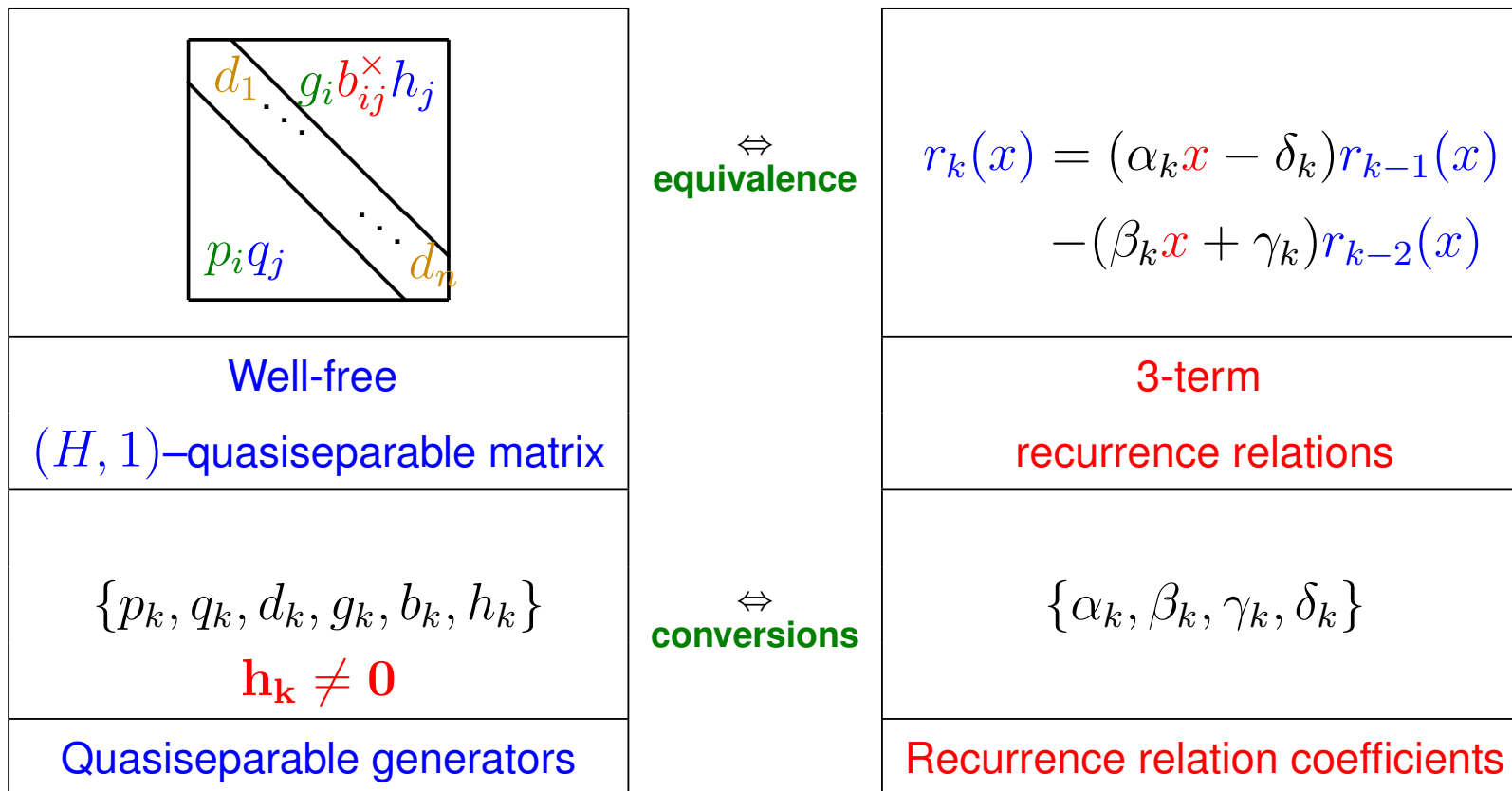
▮▮▮ Szegő polynomials (orthogonal on the unit circle): $\gamma_k = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) r_{k-2}(x)$$

The Corresponding Matrix Class: Well-Free Matrices.



Well-Free Matrices & 3-term Recurrence Relations



Subclasses of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations

Tridiagonal matrices

Unitary
Hessenberg matrices

Subclasses of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations

Well-free matrices

3-term r.r.

Tridiagonal matrices

Unitary
Hessenberg matrices

Szegő-type Two-term Recurrence Relations

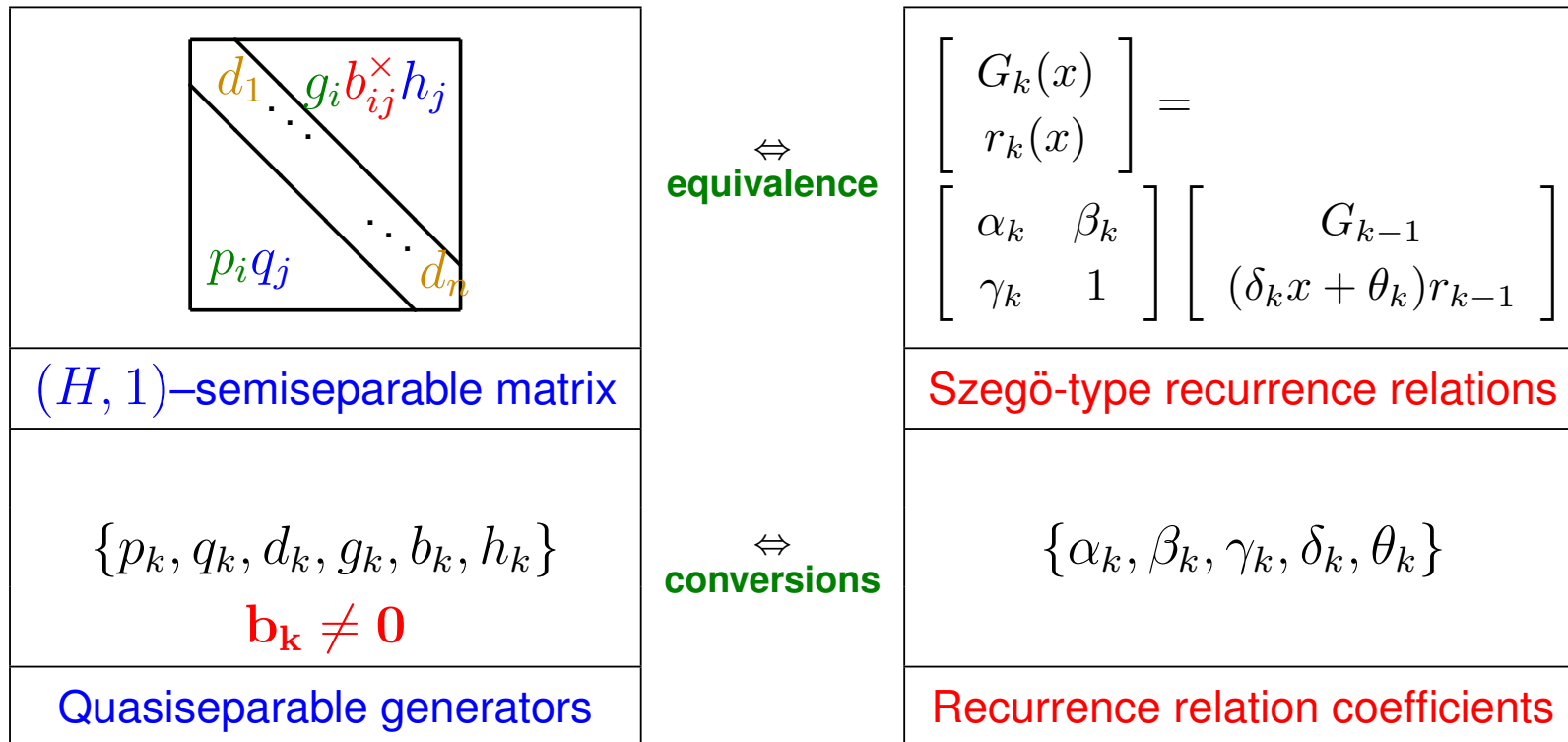
⇒ Szegő polynomials satisfy two-term recurrence relations of the form

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{x}\mathbf{r}_k(x) \end{bmatrix}.$$

⇒ Is there a class of polynomials **larger** than Szegő that satisfy two-term recurrence relations of the form

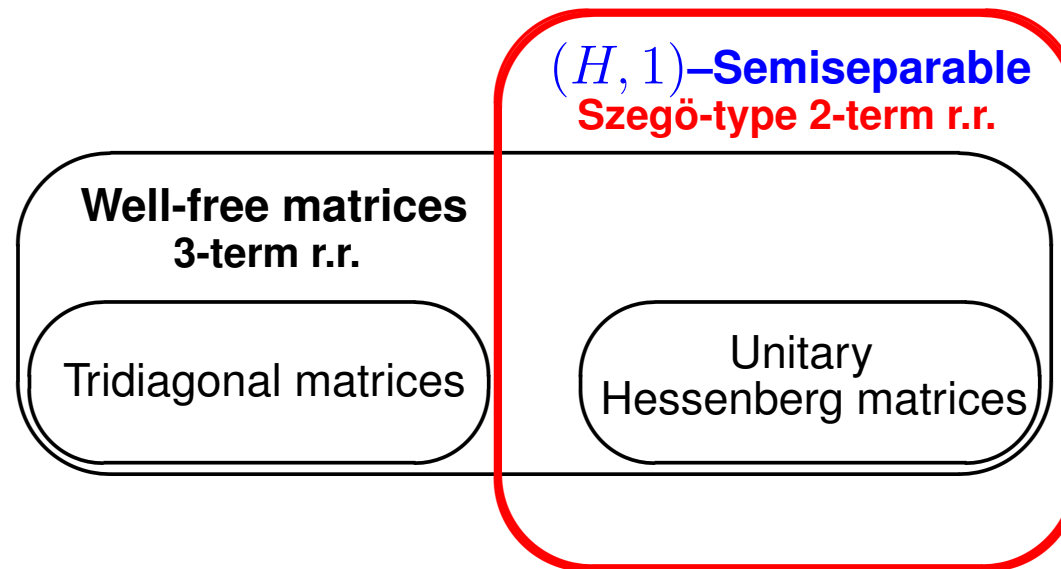
$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r}_{k+1}(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ (\delta_k x + \theta_k)\mathbf{r}_k(x) \end{bmatrix}?$$

Semiseparable Matrices & Szegő-type 2-term Recurrence Relations



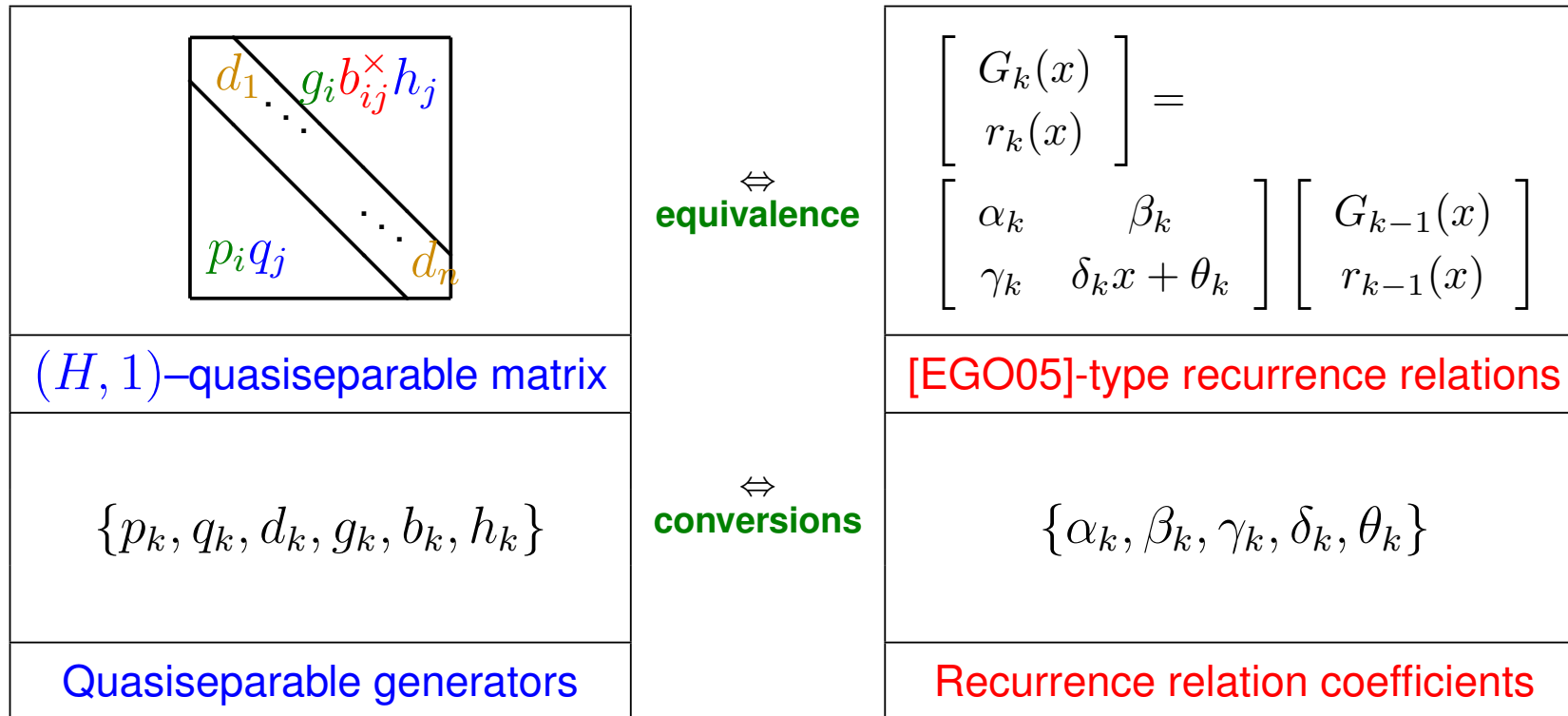
Subclasses of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations



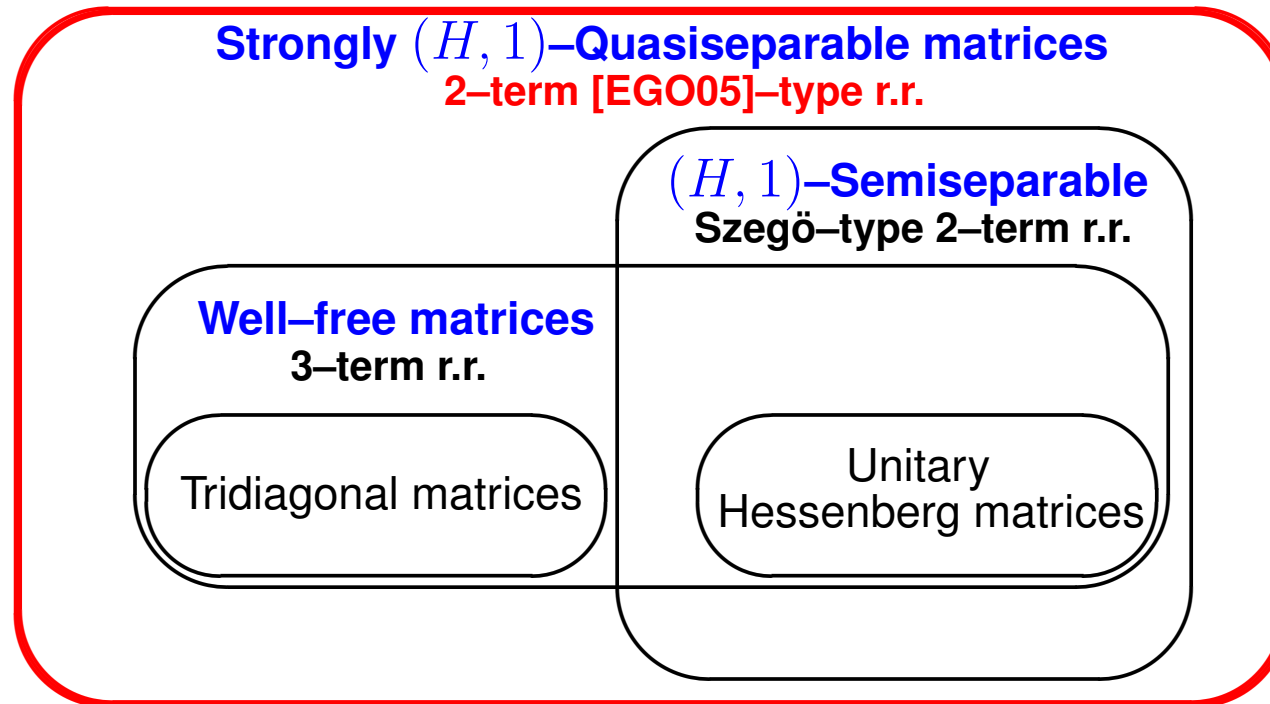
Quasiseparable Matrices & [EGO05]-type 2-term Recurrence Relations

A Complete Characterization of $(H, 1)$ -Quasiseparable Matrices



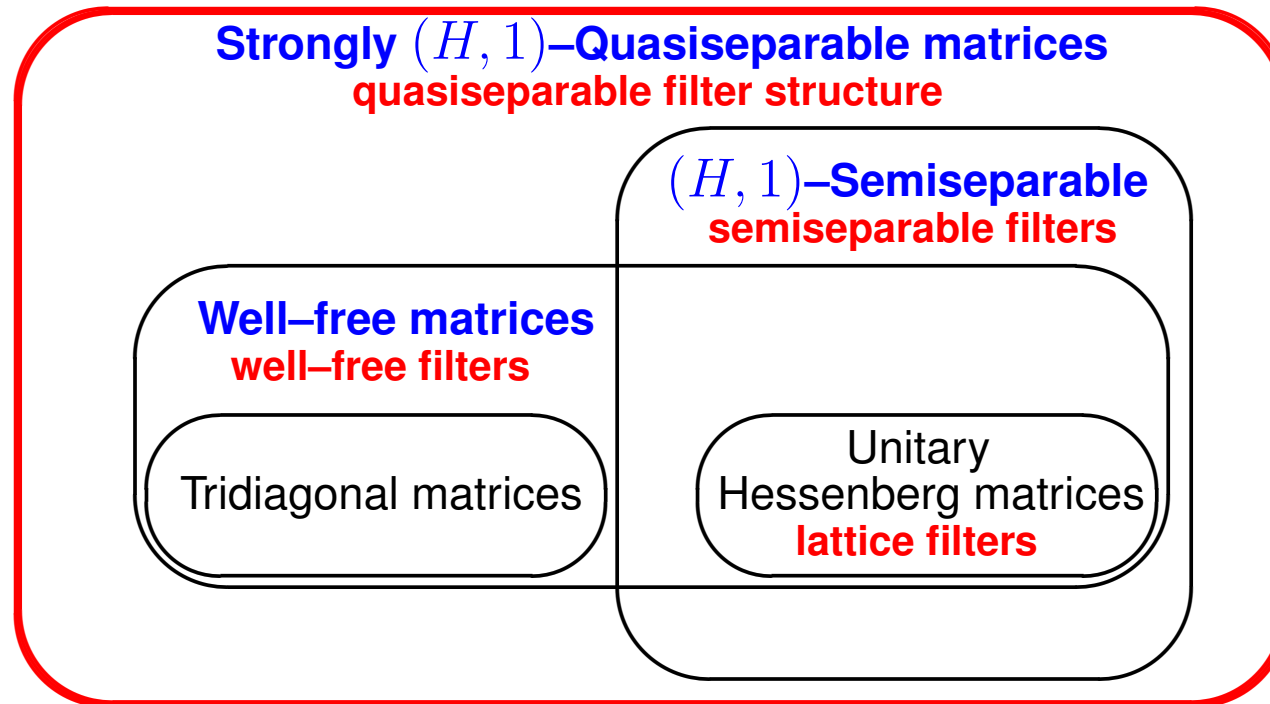
Full Characterization of $(H, 1)$ -Quasiseparable Matrices

Corresponding recurrence relations



Full Characterization of $(H, 1)$ -Quasiseparable Matrices

Corresponding digital filter structures

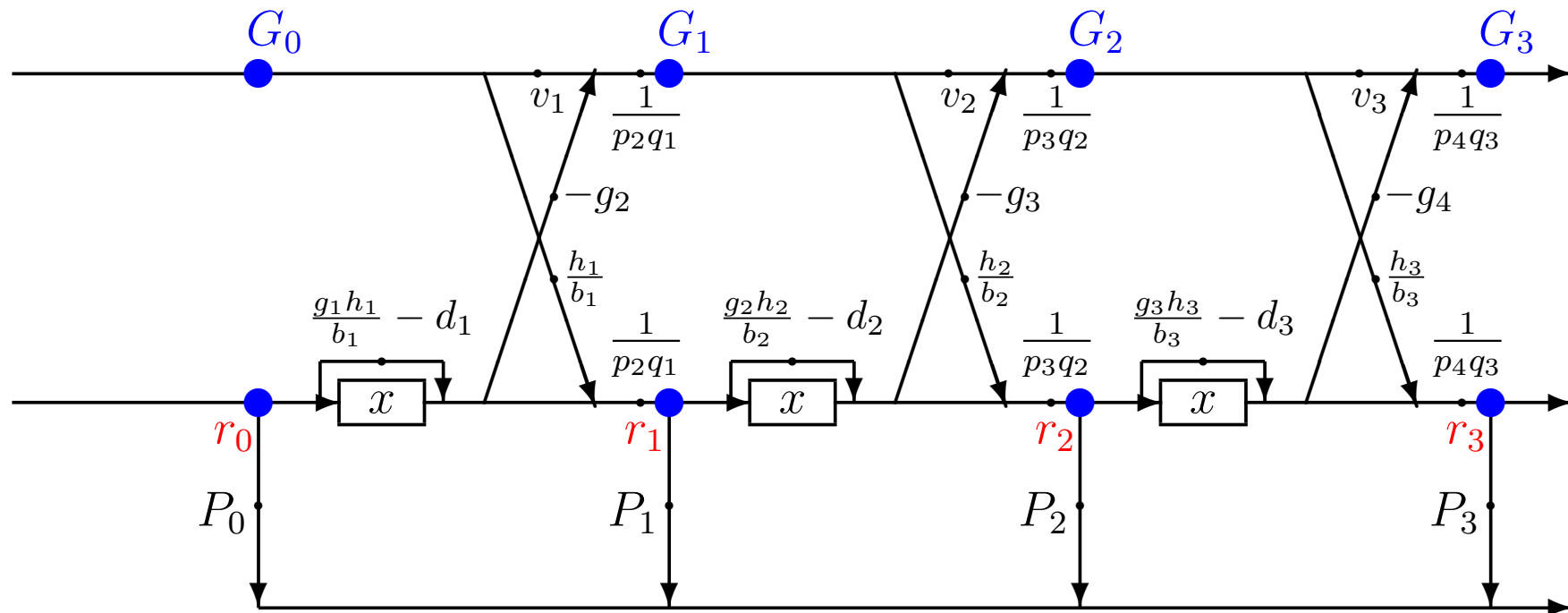


Semiseparable filter structures

➡ **Theorem.** Matrix A is $(H, 1)$ -semiseparable if and only if the polynomials

$$r_k(x) = \det(xI - A)_{(k \times k)}$$

admit the following lattice-like realization

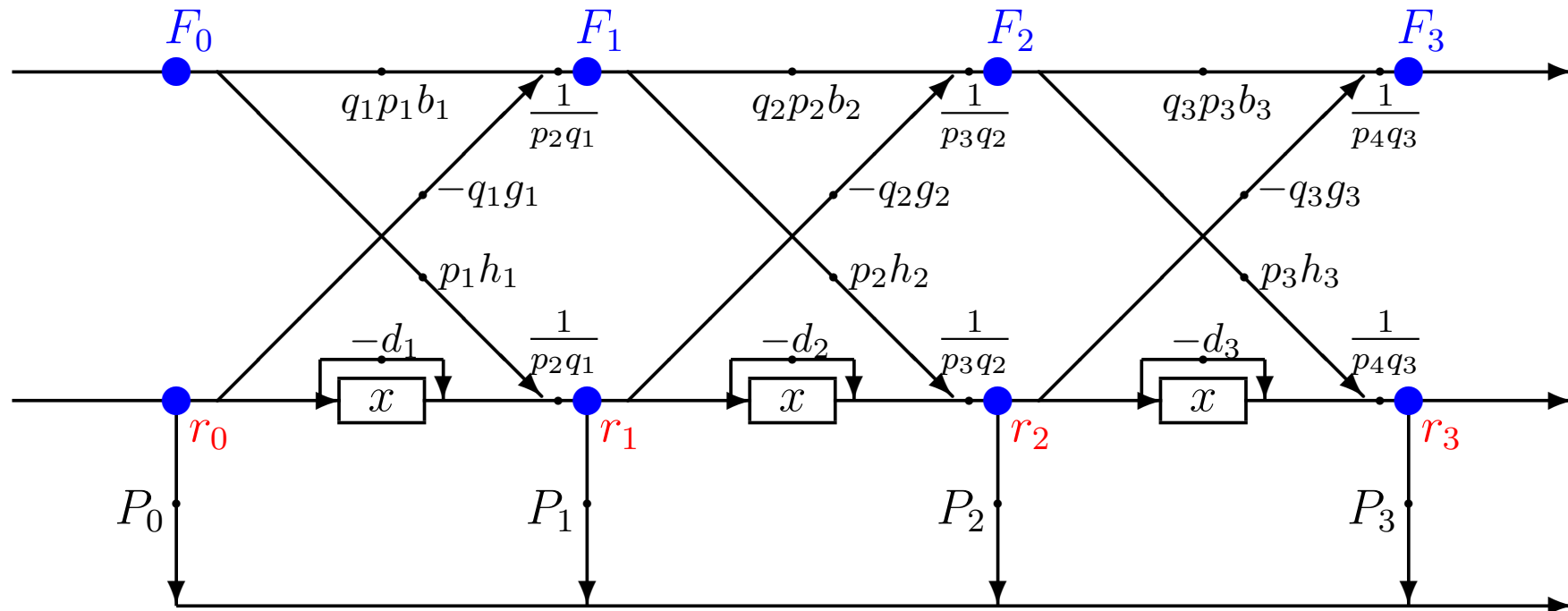


Quasiseparable filter structures

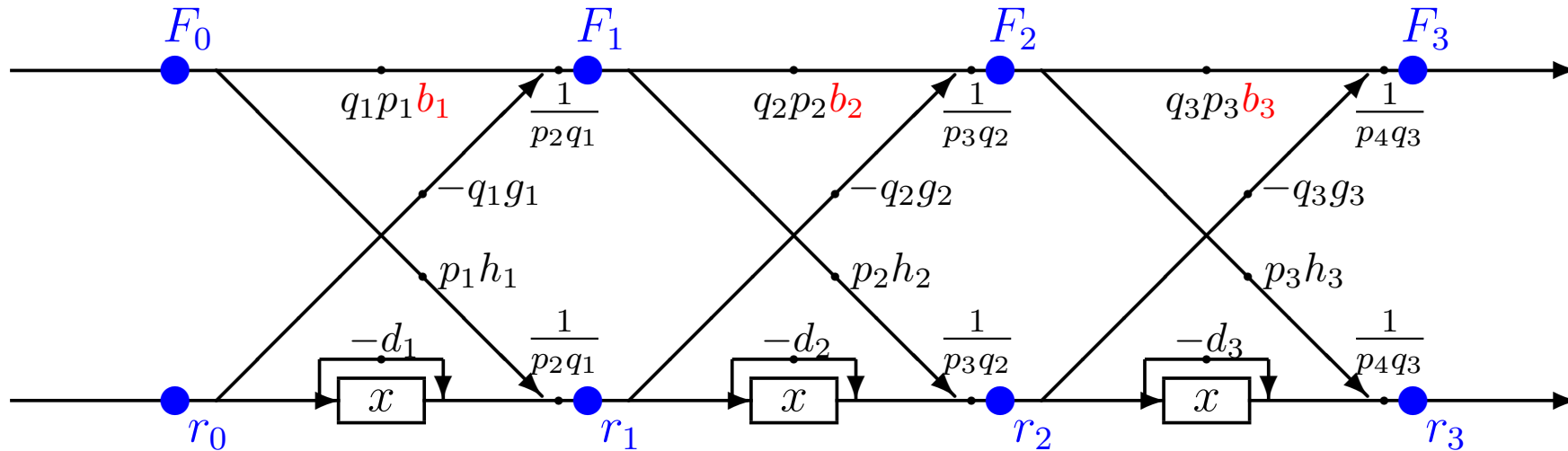
⇒ **Theorem.** Matrix A is $(H, 1)$ -quasiseparable if and only if the polynomials

$$r_k(x) = \det(xI - A)_{(k \times k)}$$

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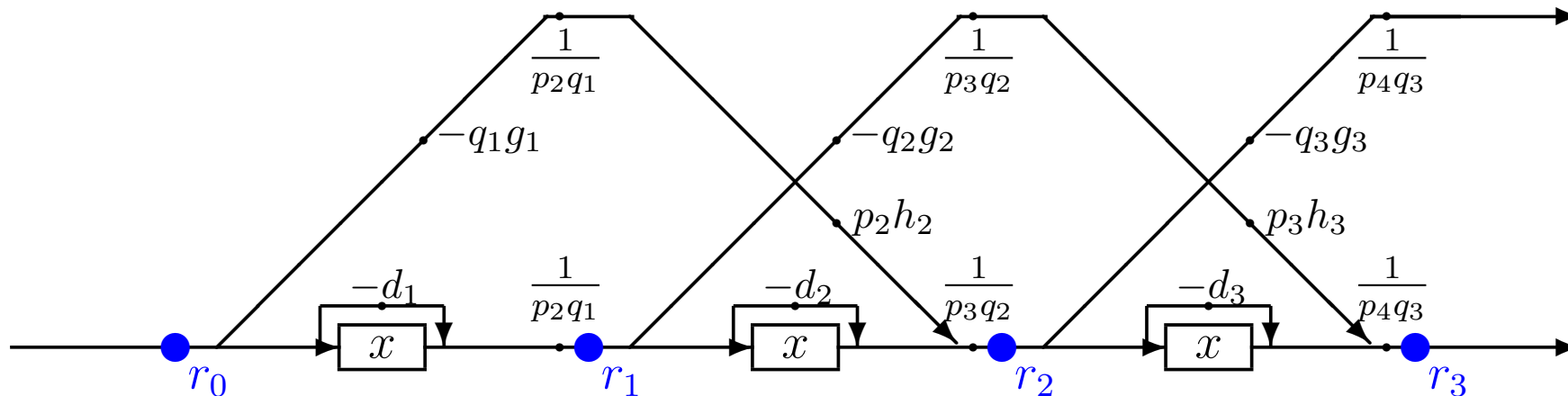


Signal flow graph for real orthogonal polynomials using quasiseparable filter structure



$$\begin{bmatrix}
 d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\
 p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\
 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\
 & & p_4 q_3 & d_4 & g_4 h_5 \\
 & & & p_5 q_4 & d_5
 \end{bmatrix}$$

Signal flow graph for real orthogonal polynomials using quasiseparable filter structure



$$\begin{bmatrix} d_1 & g_1 h_2 & & & \\ p_2 q_1 & d_2 & g_2 h_3 & & \\ & p_3 q_2 & d_3 & g_3 h_4 & \\ & & p_4 q_3 & d_4 & g_4 h_5 \\ & & & p_5 q_4 & d_5 \end{bmatrix}$$

Recurrence relation classification of (H, m) -quasiseparable matrices

Joint work with Vadim Olshevsky and Pavel Zhlobich

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A generator representation for (H, m) -quasiseparable polynomials.

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

A generator representation for (H, m) -quasiseparable polynomials.

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

$(H, 1)$ -quasiseparable generators are scalars	(H, m) -quasiseparable generators are matrices
$\boxed{g_1} \times \boxed{b_2} \times \boxed{b_3} \times \boxed{h_4}$	$\boxed{g_1} \times \boxed{b_2} \times \boxed{b_3} \times \boxed{h_4}$

What recurrence relations are satisfied by (H, m) -quasiseparable polynomials?

➡ The recurrence relations satisfied by $(H, 1)$ -quasiseparable polynomials are

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

➡ The recurrence relations satisfied by (H, m) -quasiseparable polynomials are

$$\begin{bmatrix} \boxed{F_k(x)} \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \boxed{\alpha_k} & \boxed{\beta_k} \\ \boxed{\gamma_k} & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} \boxed{F_{k-1}(x)} \\ r_{k-1}(x) \end{bmatrix}$$

What recurrence relations are satisfied by (H, m) -semiseparable polynomials?

➡ The recurrence relations satisfied by $(H, 1)$ -semiseparable polynomials are

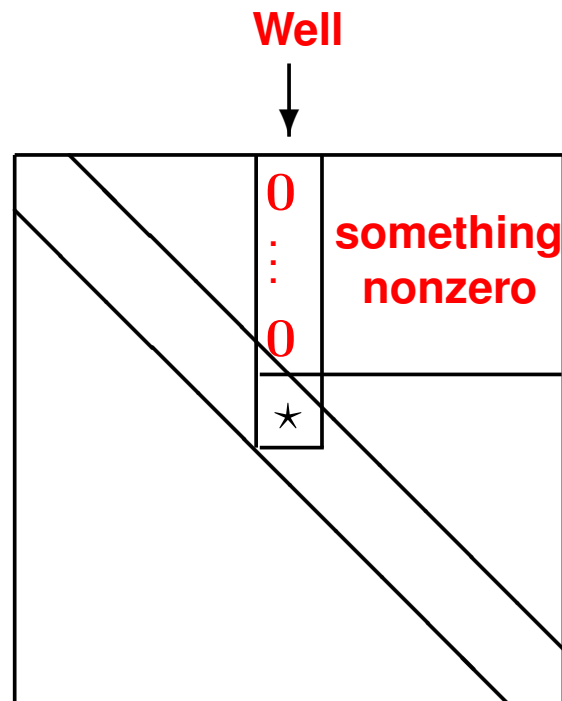
$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

➡ The recurrence relations satisfied by (H, m) -semiseparable polynomials are

$$\begin{bmatrix} G_k(x) \\ \hline r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \hline \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ \hline (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

A generalization of well-free structure?

➡ Recall that a matrix is $(H, 1)$ -well-free if it contains no “wells” of the form



➡ What is the **order** m version of this structure?

(H, m) -well-free matrices

▮ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

(H, m) -well-free matrices

▣▣▣ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▣▣▣ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

(H, m) -well-free matrices

▣▣▣ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▣▣▣ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \star & \star \end{array} \right], \quad C_{12} = \begin{array}{c} \text{rank } r_1 \\ \left[\begin{array}{ccc|ccc} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{array} \right] \end{array}$$

(H, m) -well-free matrices

▣ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▣ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

rank r_1
also rank r_1

(H, m) -well-free matrices

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$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{\text{rank } r_2}$

(H, m) -well-free matrices

▣ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▣ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \left[\begin{array}{cccccc} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{array} \right]$$

rank r_2
also rank r_2

(H, m) -well-free matrices

▣ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▣ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{\text{rank } r_3}$

(H, m) -well-free matrices

▣ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▣ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

rank r_3
rank r_3

also rank r_3

(H, m) -well-free matrices

▶▶▶▶ **Definition.** A matrix is (H, m) -well-free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

▶▶▶▶ **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

▶▶▶▶ **Example.** $m = 1$. (i.e., $(H, 1)$ -well-free)

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array} \right], \quad C_{12} = \begin{bmatrix} 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \end{bmatrix}$$

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What recurrence relations are satisfied by (H, m) -well-free polynomials?

➡ The recurrence relations satisfied by $(H, 1)$ -well-free polynomials are

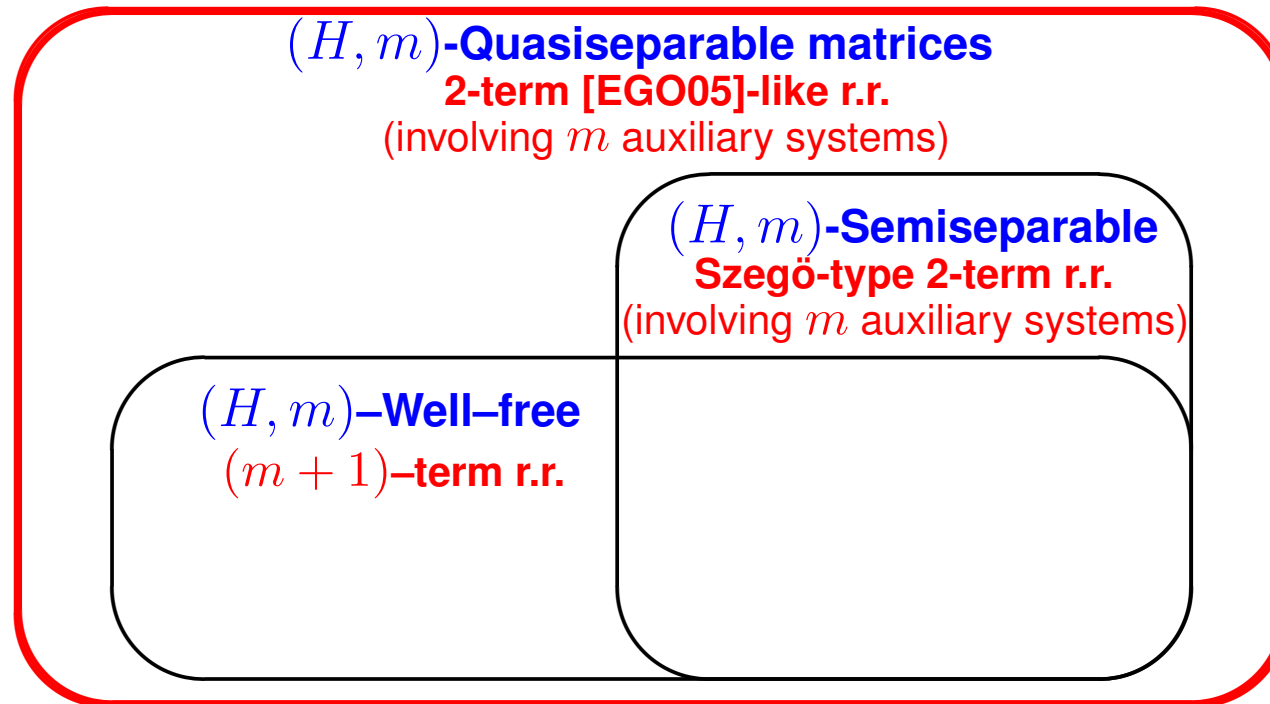
$$r_k(x) = \underbrace{(\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)}_{\text{depends on the previous two polynomials}}$$

➡ The recurrence relations satisfied by (H, m) -well-free polynomials are

$$r_k(x) = \underbrace{(\delta_{k,k} x + \epsilon_{k,k}) r_{k-1}(x) + (\delta_{k-1,k} x + \epsilon_{k-1,k}) r_{k-2}(x) + \dots}_{\text{depends on the previous } m + 1 \text{ polynomials}}$$

Full Characterization of (H, m) -quasiseparable matrices

Corresponding recurrence relations



Classifications of quasiseparable matrices in terms of recurrence relations

Tom Bella

Department of Mathematics
University of Rhode Island

Joint work with Yuli Eidelman, Israel Gohberg,
Vadim Olshevsky, & Pavel Zhlobich

